MATH 304-03 Test II Spring 2013 Name **Solutions**

Show all your work - use backs of pages if needed.

[16] 1. Let V be a vector space with $S \subseteq V$. Define the following:

(i) S is a spanning set for V, **provided every element in V is a linear combination of elements from S.**

(ii) S is a linearly independent subset of V, **provided whenever $s_1 \ldots s_n$ are distinct elements from S and $\sum_{i=1}^{n} \alpha_i s_i = 0$ then $\alpha_i = 0$ for all i.**

(iii) S is a basis for V, **S is a linearly independent spanning set for V.**

(iv) If V has a finite spanning set, dimension of V.

$\dim V$ is equal to the number of elements in a basis for V.

[18] 2. Let V be a vector space.

(i) State the principle of Independence Extension (hypothesis + conclusion).

If $I \subseteq V$, I is linearly independent, and $u \in V$ with $u \notin \operatorname{Span} I$, then $I \cup \{u\}$ is linearly independent.

(ii) State the principle of Span Preservation (hypothesis + conclusion).

If $S \subseteq V$, $u \in S$ such that $u \in \operatorname{Span}(S \setminus \{u\})$, then $\operatorname{Span}(S \setminus \{u\}) = \operatorname{Span}(S)$

(iii) Let $V = \mathbb{R}^3$, $I = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 17 \end{pmatrix} \right\}$ and $u = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

(a) Is $u \in \operatorname{Span}(I)$? **Yes** (show your work)

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 3 & 3 \\ -1 & 17 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 4 & 2 \\ 0 & -9 & -3 \\ 0 & 21 & 7 \end{pmatrix} \xrightarrow[R_3 \to \frac{1}{7} R_3]{R_2 \to -\frac{1}{3} R_2} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- column 3 is in span of columns (1) + (2).

(b) Which conclusion is valid: **X** $\operatorname{Span}\{I \cup \{u\}\} = \operatorname{Span}(I)$

or

Y $I \cup \{u\}$ is linearly independent? ? **X**

[10] 3. Let V be a vector space with a finite spanning set. Give your opinion about the following statements with True or False.

F (i) All finite spanning sets of V have the same number of elements.

T (ii) All linearly independent subsets of V have finitely many elements.

T (iii) V has a finite basis.

T (iv) All bases for V have the same number of elements.

F (v) A spanning set for V with $\dim V$ number of elements is linearly dependent.

T (vi) If W is a subspace of V with $\dim W = \dim V$, then $V = W$.

[40] 4. Let $V = P_2$, the vector space of polynomials of degree ≤ 2. As you may assume, $X = \{x^2,\ x^2+x,\ x^2+x+1\}$ and $Y = \{x^2+1,\ x+1,\ x^2-2x-2\}$ are bases for V. Considered them ordered bases ~~for~~ on this page.

(i) Calculate ${}_X(I)_Y$ (here I is the identity transformation - $Iv = v$)

Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ "the matrix for X"

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right). \text{ So } A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$y_1 = x^2+1 \to \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $A^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$y_2 = x+1 \to \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $A^{-1}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$y_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ $A^{-1}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

So ${}_XI_Y = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

(ii) Calculate ${}_Y(I)_X = ({}_XI_Y)^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -3 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right)$$

So $({}_XI_Y)^{-1} = {}_YI_X = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$.

(iii) Let $F: V \to V$ be a linear transformation for which it is known ${}_Y[F]_X = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

(a) If $v \in V$ such that the coordinates of v w.r.t. X are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = [v]_X$, calculate $F(v)$. We know $[F(v)]_Y = {}_Y[F]_X [v_X] = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

So $v = 2(x^2+1) + 3(x+1) + 1(x^2-2x-2) = 3x^2 + x + 3$

(b) Calculate ${}_X[F]_Y$

Note ${}_X[F]_Y = {}_XI_Y\ {}_YF_X\ {}_XI_Y = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 4 & 6 & 4 \\ -1 & -2 & -3 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ -2 & -2 & 3 \\ 3 & 4 & -9 \end{pmatrix}$

[38] 5. Let V be the subspace of space differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ spanned by $B = \{x\cos x,\ x\sin x,\ \cos x,\ \sin x\}$

Assume, as is true, B is a linearly independent set.

(i) $\dim V = \underline{4}$.

Let $W = \{f \in V \mid f(0) = 0\}$.

(ii) Show W is a subspace of V.

The 0 function is $O(x) = 0$ for all $x \in \mathbb{R}$. So $O \in W$.

If $f, g \in W$, $f+g\,(0) \overset{\text{def of adding functions}}{=} f(0) + g(0) = 0 + 0 = 0$. So $f+g \in W$

If $f \in W$, $a \in \mathbb{R}$ $\quad af\,(0) = a\,(f(0)) = a \cdot 0 = 0$. So $af \in W$.

(iii) Find a set of 3 linearly independent vectors in W.

$\{x\cos x,\ x\sin x,\ \sin x\} \subset W$ and is a subset of a linearly independent set, and so are linearly independent.

(iv) $\dim W = \underline{3}$. Note $\cos x \notin W$. Thus $\dim W < \dim V$

Consider $D: V \longrightarrow V$ by $D(f) = f'$ (the derivative)

(v) Calculate ${}_B(D)_B$

$$[D(x\cos x)]_B = [\cos x - x\sin x]_B = [-1(x\sin x) + 1\cos x]_B = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$[D(x\sin x)]_B = [\sin x + x\cos x]_B = [1\,(x\cos x) + 1\sin x]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[D(\cos x)]_B = [-\sin x] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$[D(\sin x)]_B = [\cos x]_B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{So } {}_B[D]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Let $D^2 = D \circ D$, the composition of functions.

(vi) Calculate ${}_B(D^2)_B$

$${}_B(D^2)_B = {}_B(DD)_B = {}_B(D)_B\,{}_B(D)_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -2 & 0 & 0 & -1 \end{pmatrix}$$

(vii) ~~Calclate~~ Calculate ${}_B[D^2]_B\,[x\cos x]_B$

$${}_B[D^2]_B\,[x\cos x]_B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -2 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

i.e. $\dfrac{d^2(x\cos x)}{dx^2} = -x\cos x - 2\sin x$

[41] 6. Let $V = P_2$, $W = \mathbb{R}^3$ and the linear transformation $F : V \to W$ by $F(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + a_1 + a_2 \\ 2a_0 - a_1 + 2a_2 \\ a_0 + a_2 \end{pmatrix}$

Assume, as it is true, $X = \{x+1,\ x^2+1,\ x^2+x+1\}$ and $Y = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ are ordered bases for V and W respectively.

(i) Calculate ${}_Y(F)_X$

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, calculation gives $A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $A^{-1} K_Y(F(x+1)) = A^{-1}\left(\begin{pmatrix}2\\1\\1\end{pmatrix}\right) = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1\end{pmatrix}$

$A^{-1}(F(x^2+1)) = A^{-1}\begin{pmatrix}2\\4\\2\end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}2\\4\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$

$A^{-1}(F(x^2+x+1)) = A^{-1}\begin{pmatrix}3\\3\\2\end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}3\\3\\2\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$

So ${}_Y(F)_X = \begin{pmatrix} 1/2 & 1 & 1 \\ 1/2 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

(ii) Find a basis for $F(V)$, the image of V under F.

$$\begin{pmatrix} 1/2 & 1 & 1 \\ 1/2 & -1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 \\ 1 & -2 & 0 \\ 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $K_Y(F(V))$ has basis $\begin{pmatrix}1/2\\1/2\\1\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}$

and so $F(V)$ has basis: $\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$ and $1\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}1\\-1\\0\end{pmatrix} + 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\4\\2\end{pmatrix}$

(iii) Find a basis for ker F.

basis for ker ${}_Y(F)_X$ is $x_1 = 2$, $x_2 = 1$, $x_3 = -2$ (free) $\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}\right\}$ is a basis for ker ${}_Y(F)_X$

$\therefore \{2(x+1) + 1(x^2+1) - 2(x^2+x+1)\} = \{-x^2+1\}$ is a basis for ker F.

(iv) Is F onto? NO

(v) Is F one-to-one? NO