This exam is worth 150 pts. Show all your work - you may use backs of page.

[30] 1. Find all solutions (if any) to the following system of equations.

$$x - y + z = 4$$
$$2x + y + 3z = 1$$
$$7x - y + 9z = 14$$

Setting up augmented matrix for the system and reducing to reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & 1 & 3 & 1 \\ 7 & -1 & 9 & 14 \end{array}\right) \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 3 & 1 & -7 \\ 0 & 6 & 2 & -14 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 3 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{1}{3} R_2]{R_1 \to R_1 + \frac{1}{3} R_2} \begin{pmatrix} 1 & 0 & 4/3 & 5/3 \\ 0 & 1 & 1/3 & -7/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution set:

$$x = 5/3 - 4/3\, z$$
$$y = -7/3 - 1/3\, z$$
$$z = z$$

Which variables are basic? x, y

Which variables are free? z

Which are the pivot positions? 1,1 and 2,2

Is $x = 2$, $y = 1$, $z = 3$ a solution? NO

For $z = 3$, $y = -10 \neq 1$

… be a set of matrices; $X=\{A,B,C,D,E\}$

where $A=\begin{pmatrix}1&-1&1\\2&0&1\end{pmatrix}$, $B=\begin{pmatrix}1&1&1\\1&2&3\\1&3&2\end{pmatrix}$, $C=\begin{pmatrix}1&3\\2&2\\3&1\end{pmatrix}$, $D=\begin{pmatrix}1&0&1&0\\2&1&2&0\\0&1&1&0\\1&1&0&0\end{pmatrix}$

and $E=\begin{pmatrix}1&2&3&2&1\\4&4&1&1&1\end{pmatrix}$.

(i) Assume one matrix in X is invertible. Which one is it? **B**

(ii) Find two matrices S, P in X with $S\neq P$ for which $3S-2P^T$ is defined.

$S=$ **A**, $P=$ **C**, $3S-2P^T=$

a. C   **A**

$$3A-2C^T=\begin{pmatrix}3&-3&3\\6&0&3\end{pmatrix}-\begin{pmatrix}2&4&6\\6&4&2\end{pmatrix}=\begin{pmatrix}1&-7&-3\\0&-4&1\end{pmatrix}$$

$$3C-2A^T=\begin{pmatrix}3&9\\6&6\\9&3\end{pmatrix}-\begin{pmatrix}2&4\\-2&0\\2&2\end{pmatrix}=\begin{pmatrix}1&5\\8&6\\7&1\end{pmatrix}$$

(iii) Find 3 matrices S, P, Q in X such that $S\neq P$ but $S+PQ$ defined.

$S=$ **B**, $P=$ **C**, $Q=$ **A**

a. $S=$ **C**, $P=$ **B**, $Q=$ **C**

$S+PQ=$

$$B+CA=\begin{pmatrix}1&1&1\\1&2&3\\1&3&2\end{pmatrix}+\begin{pmatrix}1&3\\2&2\\3&1\end{pmatrix}\begin{pmatrix}1&-1&1\\2&0&1\end{pmatrix}=\begin{pmatrix}8&0&5\\7&0&7\\6&0&6\end{pmatrix}$$

$$C+BC=\begin{pmatrix}1&3\\2&2\\3&1\end{pmatrix}+\begin{pmatrix}1&1&1\\1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&3\\2&2\\3&1\end{pmatrix}=\begin{pmatrix}7&9\\16&12\\10&12\end{pmatrix}$$

(iv) Find matrices S, P in X such that SP is a 3×5 matrix

$S=$ **C**

$P=$ **E**

$$SP=\begin{pmatrix}1&3\\2&2\\3&1\end{pmatrix}\begin{pmatrix}1&2&3&2&1\\4&4&1&1&1\end{pmatrix}=\begin{pmatrix}13&14&6&5&4\\10&12&8&6&4\\7&10&10&7&4\end{pmatrix}$$

$$P^TS^T=(SP)^T=\begin{pmatrix}13&10&7\\14&12&10\\6&8&10\\5&6&7\\ \end{pmatrix}$$

3. (i) Find the matrix E such that if A is a $4\times n$ matrix then EA is the matrix obtained by performing the elementary row operation of replacing row 2 by row 2 minus 3 times row 3.

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Let $A = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$. Write A and A^{-1} as a product of elementary matrices.

$$\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} \xrightarrow{R_1 \to -R_1} \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ & \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ & \end{pmatrix} = E\,E\,E\,E\,E.$$

... Suppose V, W are vector spaces and $T : V \to W$ is a function.

(i) domain of T = $\underline{V}$ (ii) codomain of T = $\underline{W}$

(iii) What is the definition of "T is a linear transformation"

For $u, v \in V$, $T(u+v) = T(u) + T(v)$
and for $v \in V$, $c \in \mathbb{R}$, $T(cv) = cT(v)$

(iv) Now suppose $V = \mathbb{R}^3$, $W = \mathbb{R}^4$, T is a linear transformation, and it is known $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$, $T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ and $T = \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Write the standard matrix for T

Note $\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}$ So $T(e_2) = T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}$ So $T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\begin{pmatrix}1\\1\\1\end{pmatrix} - T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}$

Standard matrix for T is $\begin{pmatrix}0 & 1 & -1\\1 & 0 & -1\\0 & 0 & 0\\1 & -1 & 1\end{pmatrix}$

(v) For T as given in (iv), calculate $T\begin{pmatrix}2\\-1\\4\end{pmatrix}$.

$$T\begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\\1 & 0 & -1\\0 & 0 & 0\\1 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}-5\\-2\\0\\7\end{pmatrix}$$

[10] 5. (i) What is the definition of "U is a subspace of vector space V"?

U is a subset of V which is a vector space under the restricted operations on V. | or $U \subseteq V$, $U \neq \emptyset$, if $u, w \in U$, then $u+w \in U$; if $u \in V$, $c \in \mathbb{R}$ then $cu \in U$

(ii) True or False : $\underline{F}$ $\{(u, v) \mid u, v$ are even integers$\}$ is a subspace of $\mathbb{R}^2$.

$\underline{T}$ $\{(u, v, w) \mid u + v = -w\}$ is a subspace of $\mathbb{R}^3$

$\underline{T}$ If U is a subspace of V and L is a linear transformati...

L-6. Assume to row reduction to reduced echelon form is correc

$$\left(\begin{array}{ccccc|c} 1 & -2 & -3 & 0 & 1 & a \\ 0 & 1 & 2 & 1 & 0 & b \\ 2 & 0 & 3 & 1 & 1 & c \\ 5 & 0 & 7 & 4 & 3 & d \\ 0 & 0 & 0 & 0 & 1 & e \end{array}\right) \text{ reduces to } \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 3a+6b-c-2e \\ 0 & 1 & 0 & 7 & 0 & 4a+9b-2c-2e \\ 0 & 0 & 1 & -3 & 0 & -2a-4b+c+e \\ 0 & 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 & -a-2b-2c+d \end{array}\right)$$

C is the coefficient matrix.

(i) Find a vector x (if it exists), $x \in \mathbb{R}^5$ with $Cx = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

Since $-1-2-2+2 \neq 0$

there is no such x.

(ii) Find a vector y (if it exists) $y \in \mathbb{R}^5$ with $Cy = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 5 \\ 2 \end{pmatrix}$

There is such a y - y_4 is free - let $y_4 = 0$

$y_1 = 3+6-1-4 = 4$

$y_2 = 4+9-2-4 = 7$

$y_3 = -2-4+1+2 = -3$

$y_5 = 2$

So $y = \begin{pmatrix} 4 \\ 7 \\ -3 \\ 0 \\ 2 \end{pmatrix}$ is such a vector

(iii) Find (if it exists) a nonzero vector in the null space of C.

Letting $a=b=c=d=e=0$ and $y_4 = 1$

$y_1 = -5$

$y_2 = -7$

$y_3 = 3$

$y_5 = 0$

So $\begin{pmatrix} -5 \\ -7 \\ 3 \\ 1 \\ 0 \end{pmatrix}$ is in null space of C

(iv) Express (if possible) $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \\ 0 \end{pmatrix}$ as a linear combination of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 3 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} \cancel{0} \\ 0 \\ \cancel{0} \\ \cancel{0}3 \\ 1 \end{pmatrix} \right\}$

Using the same elementary row operation

$$\left(\begin{array}{cccc|c} 1 & -2 & -3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 5 & 0 & 7 & 3 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \text{ reduces to } \begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \\ 0 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 2 \\ 5 \\ 0 \end{pmatrix} + 7\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 2 \\ 3 \\ 7 \\ 0 \end{pmatrix}$