Math 220 - Calculus f. Business and Mgmt - Worksheet 41-42

Solutions for Worksheet 41-42 - Multivariable Functions and Partial Derivatives

Exercise 1:

The cost to produce products A and B is $C(a,b) = 3000 + 5a + 6b^2$. If the current production is 10 of a and 20 of b, answer the following questions:

$1a)$ What is the current cost?

$1b)$ How much will cost increase by making 1 more of product A?

$1c)$ How much will cost increase by making 1 more of product B?

Solution to #1:

Solution to #1a: Current cost (for $a = 10$ and $b = 20$) is

$$C(10,20) \ = 3,000 + 5 \cdot 10 + 6 \cdot 400 \ = \ \boxed{5,450}$$

There are two ways to interpret questions b and c:

First interpretation: We look at the differences $C(11,20) - C(10,20)$ and $C(11,20) - C(10,20)$:

First solution to #1b: Cost increase in a is

$$\begin{aligned} C(11,20) - C(10,20) \ &= 3,000 + 5 \cdot 11 + 6 \cdot 400 \ - \ 3,000 - 5 \cdot 10 - 6 \cdot 400 \\ &= 5 \cdot 11 - 5 \cdot 10 \ = \ \boxed{5} \end{aligned}$$

First solution to #1c: Cost increase in b is

$$\begin{aligned} C(10,21) - C(10,20) \ &= 3,000 + 5 \cdot 10 + 6 \cdot 21 \cdot 21 \ - \ 3,000 - 5 \cdot 10 - 6 \cdot 20 \cdot 20 \\ &= 6(441 - 400) \ = \ \boxed{246} \end{aligned}$$

Second interpretation: We look at the instantaneous rates of change with respect to A and B, i.e., $C_a(10,20)$ and $C_b(10,20)$:

Second solution to #1b: Cost increase in a is

$$C_a(a,b) \ = \frac{\partial C}{\partial a}(a,b) \ = \ 0 + 5 + 0 \ \Rightarrow \ C_a(10,20) \ = \ \boxed{5}$$

Second solution to #1c: Cost increase in b is

$$C_b(a,b) \ = \frac{\partial C}{\partial b}(a,b) \ = \ 0 + 0 + 12b \ \Rightarrow \ C_b(10,20) \ = \ 12 \cdot 20 \ = \ \boxed{240}$$

Note: If you look back at exercises 1 and 3 in worksheet 19 (using the marginals to approximate a change in function values) then you see that wks 19, #1 corresponds to #1b of the current exercise: the function is linear in the variable under consideration, hence the approximation using the marginals is exact. On the other hand, wks 19, #3c corresponds to #1c of the current exercise: the function is **NOT** linear in the variable under consideration, hence the approximation using the marginals has an error.

Exercise 2:

Find and plot the domains of these functions:

$$\textbf{2a.}\ x/y, \quad \textbf{2b.}\ x^2\sqrt{y}, \quad \textbf{2c.}\ (3x+4y)/(2x-5y), \quad \textbf{2d.}\ \sqrt{x+y}, \quad \textbf{2e.}\ \ln(xy-1).$$

Solution to #2a: We are not allowed to divide by zero. Hence $D_f \;=\; \{(x,y) : -\infty < x < \infty, y \neq 0\}$. (all points not on the x-axis)

Solution to #2b: Square roots must be non-negative. Hence $D_f \;=\; \{(x,y) : -\infty < x < \infty, y \geqq 0\}$. (all points above and including the x-axis)

Solution to #2c: We must have $2x - 5y \neq 0$. Hence $D_f \;=\; \{(x,y) : -\infty < x < \infty, y \neq \dfrac{2x}{5}\}$.

Note that $D_f \;=\; \{(x,y) : -\infty < y < \infty, x \neq \dfrac{5y}{2}\}$.

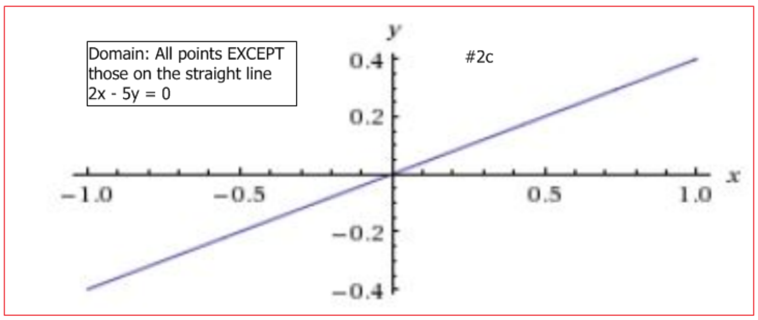

Figure 1: Domain for #2c

Solution to #2d: We must have $x + y \geqq 0$. Hence $D_f \;=\; \{(x,y) : -\infty < x < \infty, y \geqq -x$.
Note that $D_f \;=\; \{(x,y) : -\infty < y < \infty, x \geqq -y\,\}$.

Solution to #2e: We must have $xy - 1 > 0$. Hence

$$D_f \;=\; \{(x,y) : xy > 1\} \;=\; \{(x,y) : x > 0 \text{ and } y > \frac{1}{x}\} \bigcup \{(x,y) : x < 0 \text{ and } y < \frac{1}{x}\}$$

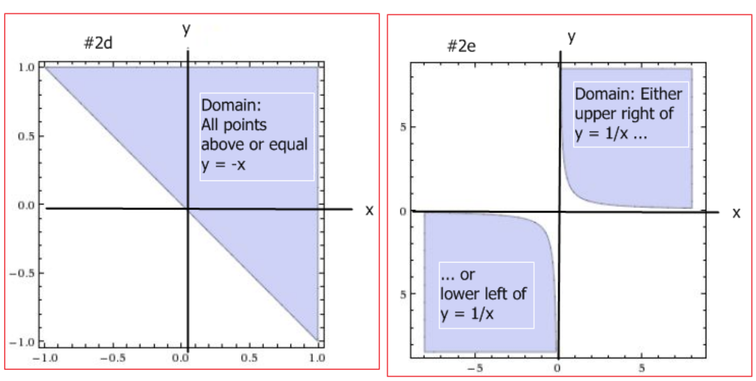

Figure 2: Domains for #2d, #2e

Exercise 3:

Find the first derivatives of the cost equation in problem 1. Evaluate the derivatives at $a = 10$ and $b = 20$. Compare your answers to the answers in parts b and c of problem 1.

Solution to #3:

$$\begin{aligned} C(a,b) &= 3000 + 5a + 6b^2 \rightsquigarrow C_a(a,b) = \frac{\partial C}{\partial a}(a,b) = 5 \rightsquigarrow C_a(10,20) = 5 \\ C_b(a,b) &= \frac{\partial C}{\partial b}(a,b) = 12b \rightsquigarrow C_b(10,20) = 240 \end{aligned}$$

We note that there is a close relationship between the change in cost for an additional unit and the derivative with respect to that type of unit but not exact equality.

Exercise 4:

Find the partial derivatives f_x and f_y of the following functions:

4a. $f(x,y) = 2x^2 + 4xy^3 - 6y^2$, **4b.** $f(x,y) = e^x \ln(y)$, **4c.** $f(x,y) = (3x^2 - 5xy)(4\ln x)$,
4d. $f(x,y) = (6e^y)(3xy^4 - 2x^2y^2)$, **4e.** $f(x,y) = \dfrac{3x^2y^2 - 6x^3}{5xy + 4y}$, **4f.** $f(x,y) = e^{5x^2 - 3y + 7}$,
4g. $f(x,y) = \ln(5xy + 3x^2 - 2y^4)$.

Solution to #4a:

$$f_x = 4x + 4y^2; \quad f_y = 12xy^2 - 12y.$$

Solution to #4b:

$$f_x = e^x \ln(y); \quad f_y = \frac{e^x}{y}.$$

Solution to #4c: The product rule becomes $(uv)_x = u_x v + u v_x$ for $\partial/\partial x$:

$$\begin{aligned} f_x &= (6x - 5y)(4\ln x) + (3x^2 - 5xy)\frac{4}{x} = 4(6x - 5y)\cdot \ln x + 4(3x - 5y); \\ f_y &= (-5y)(4\ln x). \end{aligned}$$

Solution to #4d: We use the product rule to compute f_y:

$$f_x = 6e^y \cdot (3y^4 - 4xy^2); \quad f_y = 6e^y(3y^4 - 4xy^2) + 6e^y(12xy^3 - 4x^2y).$$

Solution to #4e: The quotient rule becomes $(u/v)_x = (u_x v - u v_x)/v^2$ for $\partial/\partial x$ and $(u/v)_y = (u_y v - u v_y)/v^2$ for $\partial/\partial y$:

$$\begin{aligned} f_x &= \frac{(6xy^2 - 18x^2)\cdot(5xy + 4y) - (3x^2y^2 - 6x^3)\cdot(5y)}{(5xy + 4y)^2} \\ f_y &= \frac{6x^2y)\cdot(5xy + 4y) - (3x^2y^2 - 6x^3)\cdot(5x + 4)}{(5xy + 4y)^2} \end{aligned}$$

Solution to #4f: We use the chain rule for both f_x and f_y:

$$f_x \ = e^{5x^2-3y+7}10x; \quad f_y \ = \ e^{5x^2-3y+7}(-3).$$

Solution to #4g: We use the chain rule for both f_x and f_y:

$$f_x \ = \frac{5y+6x}{5xy+3x^2-2y^4}; \quad f_y \ = \ \frac{5x-8y^3}{5xy+3x^2-2y^4}.$$

Exercise 5:

Find all four second derivatives of the following functions:

$$\textbf{5a.}\ f(x) \ = 5x^7y^3, \quad \textbf{5b.}\ f(x) \ = \ \ln(6x^2-2y^3).$$

Solution to #5a:

$$\begin{aligned} f_x \ &= 35x^6y^3; \quad f_{xx} \ = \ 210x^5y^3; \quad f_{xy} \ = \ 105x^6y^2; \\ f_y \ &= 15x^7y^2; f_{yx} \ = \ 105x^6y^2; \quad f_{yy} \ = \ 30x^7y; \end{aligned}$$

We note that $f_{xy} = f_{yx}$.

Solution to #5b: Only the chain rule is needed for f_x and f_y:

$$\begin{aligned} f_x \ &= \frac{12x}{6x^2-2y^3} \ = \ (12x)\,(6x^2-2y^3)^{-1} \\ f_y \ &= \frac{-\,6y^2}{6x^2-2y^3} \ = \ (-6y^2)\,(6x^2-2y^3)^{-1}; \end{aligned}$$

We use the quotient rule as we did in #4e for f_{xx} and f_{yy}:

$$\begin{aligned} f_{xx} \ &= \frac{12(6x^2-2y^3) \ - \ (12x)(12x)}{(6x^2-2y^3)^2} \ = \ \frac{12(6x^2-2y^3) \ - \ 144x^2}{(6x^2-2y^3)^2} \\ f_{yy} \ &= \frac{(-12y)(6x^2-2y^3) \ - \ (12x)(-6y^2)}{(6x^2-2y^3)^2}; \end{aligned}$$

The mixed partials f_{xy} and f_{yx} do NOT need the quotient rule because $(12x)$ is constant in y and $(-6y^2)$ is constant in x:

$$\begin{aligned} f_{xy} \ &= (12x)\,(-1)(6x^2-2y^3)^{-2}(-6y^2) \\ f_{yx} \ &= (-6y^2)\,(-1)(6x^2-2y^3)^{-2}(12x); \end{aligned}$$

We note that $f_{xy} = f_{yx}$.