Math 220 - Calculus f. Business and Management - Worksheets 23 - 25

Solutions for Worksheets 23-25 - Graphing Functions

This worksheet covers lessons 23-25. Given are several functions which you are asked to graph. You will do so in three stages, as the graphing of all features requires that you are familiar with the material of all three lessons.

Class 23: Using the function

Stage 1: *You determine the features which do not involve first or second derivatives: domain, x-intercept, y-intercept (if not too difficult), horizontal asymptotes, points of discontinuity (vertical asymptotes, jump discontinuities, "holes").*

Class 24: Using first and second derivatives

Stage 2a: *You use the first derivative to compute the critical points and you determine whether it is positive or negative between two adjacent CP's. This lets you determine whether $f(x)$ increases or decreases on such intervals.*

Stage 2b: *You then use the second derivative to compute the points x for which $f''(x) = 0$ or $f''(x)$ DNE. and you determine whether it is positive or negative between two adjacent such points. This lets you determine whether $f(x)$ is concave up (curves to the left as x moves in positive direction) or concave down (curves to the right as x moves in positive direction) on such intervals.*

Class 25: Sketching the graphs

Stage 3: *You use what you have found out in the previous stages to graph the function.*

Exercise 1: *Graph the function* $f(x) = 3x^3 - 2x + 7$

Solution to #1:

#1 - Stage 1*:*

Domain: all real numbers as $f(x)$ is a polynomial.
y-intercept: $f(0) = 7$
x-intercept: too difficult to compute an exact value but we see that there is a zero between $x = -2$ and $x = -1$ as $f(x)$ switches sign: $f(-2) = -18 - 4 + 7 < 0$ and $f(-1) = -3 - 2 + 7 > 0$.
Horizontal asymptotes: None: No polynomial has horizontal asymptotes unless it is of the form $f(x) = const$.
Behavior as $x \to \pm\infty$: $f(x) \approx 3x^3$ for huge $|x|$. Hence $\lim\limits_{x\to-\infty} f(x) = -\infty\,,\quad \lim\limits_{x\to\infty} f(x) = \infty\,.$
Vertical asymptotes: None and no other kind of discontinuity as $f(x)$ is a polynomial.

#1 - Stage 2a*:*

$f'(x) = 9x^2 - 2$
Zeroes: $9x^2 - 2 = 0 \iff 3x = \pm\sqrt{2}$, i.e., $x = \pm\sqrt{2}/3$.
Critical points: None besides $x = \pm\sqrt{2}/3$ as there are no "problem points" for the polynomial $f'(x)$. To check for plus or minus in each of the three intervals separated by the critical points we pick one convenient point in each. As $\sqrt{2} \approx 1.41$, we have $0 < \sqrt{2}/3 < 1$ and we can pick $x = -1, 0, 1$.

$$x = -1 : f'(x) = 9 - 2\boxed{> 0} \qquad x = 0 : f'(x) = -2\boxed{< 0} \qquad x = 1 : f'(x) = 9 - 2\boxed{> 0}$$

#1 - Stage 2b*:*

$f''(x) = 18x$
Zeroes: $18x = 0 \iff x = 0$.
Points for which $f''(x)$ DNE: None because there are no "problem points for the polynomial $f''(x)$. To check for plus or

minus in each of the two intervals separated by $x = 0$ we pick one convenient point in each. We pick $x = \pm 1$.

$$x \;=\; -1\;:\; f''(x) \;=\; -18\boxed{<0} \qquad x \;=\; 0\;:\; f''(x) \;=\; 18\boxed{>0}$$

***#1 - Stage 3**:*

The points x for which $f'(x) = 0$ or $f''(x) = 0$ split the domain $D_f = (-\infty, \infty)$ into four intervals:

$$\begin{aligned}
x \;<\; -\sqrt{2}/3\;: &\quad f'(x) =' +', \quad f''(x) \;=\; '-'\;: \quad \textit{increasing and CD (curving right)}\\
-\sqrt{2}/3 < x \;<\; 0\;: &\quad f'(x) =' -', \quad f''(x) \;=\; '-'\;: \quad \textit{decreasing and CD (curving right)}\\
0 < x \;<\; \sqrt{2}/3\;: &\quad f'(x) =' -', \quad f''(x) \;=\; '+'\;: \quad \textit{decreasing and CU (curving left)}\\
x \;>\; \sqrt{2}/3\;: &\quad f'(x) =' +', \quad f''(x) \;=\; '+'\;: \quad \textit{increasing and CU (curving left)}
\end{aligned}$$

It follows that $x = 0$ is a point of inflection as curvature changes from right to left, the critical point $-\sqrt{2}/3$ has a local max and the critical point $\sqrt{2}/3$ has a local min.

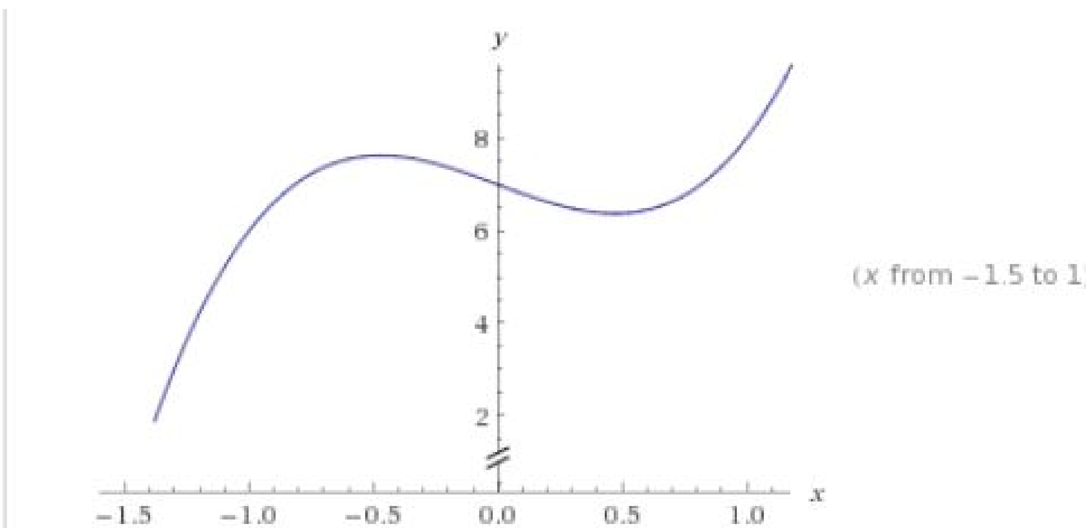

Figure 1: Graph of $f(x) \;=\; 3x^3 - 2x + 7$

***Exercise 2:** Graph the function $f(x) \;=\; e^{x^5+4x}$. Do NOT attempt to find the zeroes of $f''(x)$. It takes more skills than required for this course to figure out that there are none.*

Solution to #2:

***#2 - Stage 1**:*

Domain: all real numbers as the exponent of $f(x)$ is the composition of an exponential function with a polynomial.
y-intercept: $f(0) = e^0 = 1$
x-intercept: NONE as e^z for all numbers z.
Behavior as $x \to \pm\infty$: $x^5 + 4x \approx x^5$ for huge $|x|$. Hence $\lim\limits_{x\to-\infty} x^5 + 4x \;=\; -\infty\,,\quad \lim\limits_{x\to\infty} x^5 + 4x \;=\; \infty$. Together with the fact that $e^z \to 0$ as $z \to -\infty$ and $e^z \to \infty$ as $z \to \infty$:

$$\lim_{x\to-\infty} f(x) = 0 \qquad \lim_{x\to\infty} f(x) \;=\; \infty$$

Horizontal asymptotes: $y = 0$ *is approached as* $x \to -\infty$.
Vertical asymptotes: NONE and no other kind of discontinuity as $f(x)$ *is an exponentiated polynomial.*

#2 - Stage 2a*:*

Chain rule: $f'(x) = (5x^4 + 4)\,e^{x^5+4x}$
Zeroes: NONE because $5x^4 + 4 \geq 4 > 0$ *and* $e^{any} > 0$*, hence their product* $f'(x) > 0$
Critical points: NONE as there are no "problem points" for $f'(x)$ *as the product of a polynomial and an exponential of another polynomial.*

We remember this for later: $f'(x) > 0$ *and hence* $f(x)$ *is strictly increasing for all real numbers.*

#2 - Stage 2b*:*

$f''(x) = e^{x^5+4x}(25x^8 + 40x^4 + 20x^3 + 16)$
Zeroes: As $e^{any} > 0$ *this means finding* x *such that* $25x^8 + 40x^4 + 20x^3 + 16 = 0$*. It is way above the skills taught in this class to figure out that no such* x *can be found, hence the continuous* $f''(x)$ *is either completely above or below the* x*-axis. That's easy to figure: plug in* $x = 0$ *and you get* $f''(0) = e^1(16) = 16\,e > 0$

Here is the graph of $25x^8 + 40x^4 + 20x^3 + 16$.

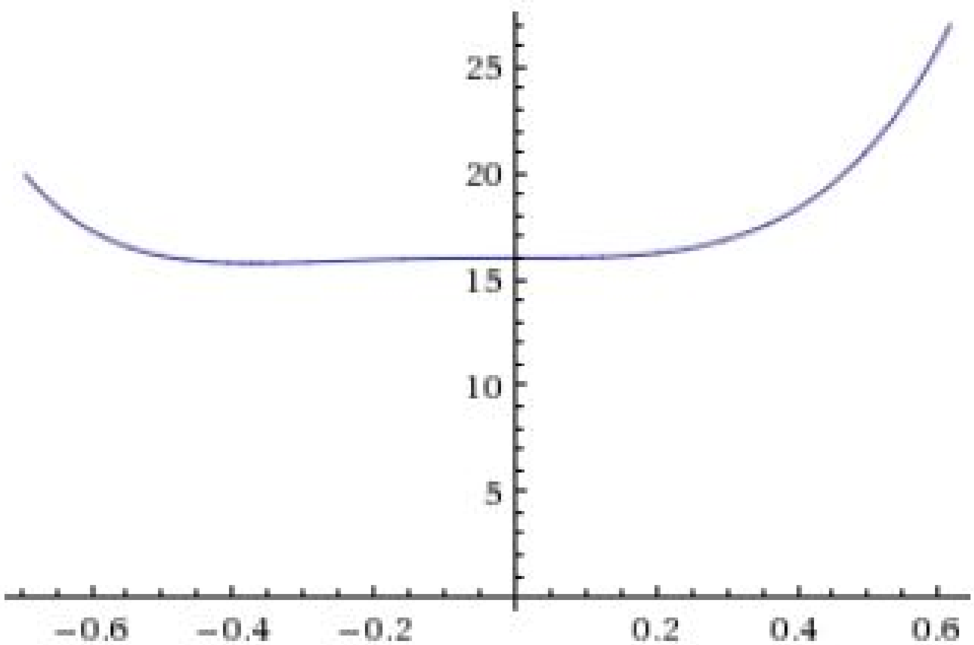

Figure 2: Graph of $25x^8 + 40x^4 + 20x^3 + 16$

We remember this for later: $f''(x) > 0$ *and hence* $f(x)$ *is CU (curving left) for all real numbers.*

#2 - Stage 3*:*

Very boring function: we found out in stage 1 that it is strictly increasing and curving left for all x*. Note that the* x*-axis hits the* y *axis not at* $x = 0$ *but, rather,* $x \approx .5$.

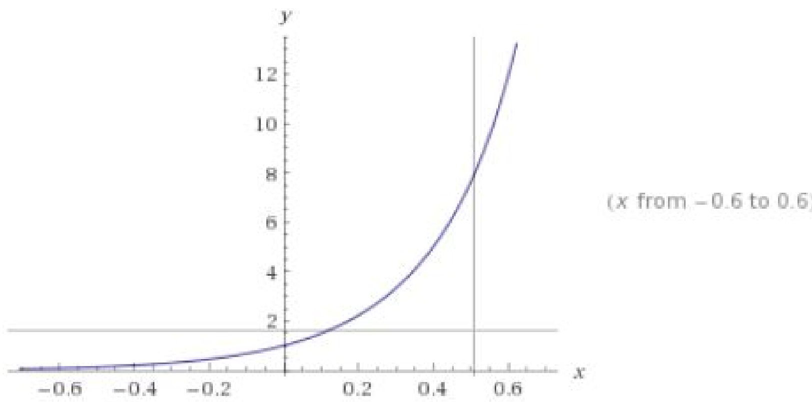

Figure 3: Graph of $f(x) = e^{x^5+4x}$

Exercise 3: *Graph the function* $f(x) = \sqrt{x-2}$

Solution to #3:

This is simply the graph of the the function $g(x) = \sqrt{x}$ *shifted 2 units to the right. You know the graph of* $\sqrt{x}$ *and that the following properties are inherited by* $f(x)$*:*
a) it strictly increases on its entire domain: expect $f'(x) \geq 0$
b) it curves to the right on its entire domain: expect $f''(x) \leq 0$

#3 - Stage 1*:*

Domain: $D_f = [2, \infty)$
y*-intercepts: NONE:* $f(0)$ *DNE*
x*-intercepts:* $\sqrt{x-2} = 0 \iff x-2=0 \iff x=2.$
Behavior as $x \to -\infty$*: Not applicable as the domain is bounded away from* $-\infty$*.*
Behavior as $x \to \infty$*:* $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \sqrt{x} = \infty$ *because* $x - 2 \approx x$ *for huge* x*. Horizontal asymptotes: NONE.*
Vertical asymptotes: NONE: The graph of $f(x)$ *gets closer and closer to the line* $x = 2$ *as* $x \to 2+$ *but the function does not blow up to* ∞ *or* $-\infty$*. The text is not explicit about this but you find in in Wikipedia (http://en.wikipedia.org/wiki/Asymptote).*

#3 - Stage 2a*:*

Chain rule: $f'(x) = 1 \cdot \dfrac{(x-2)^{-1/2}}{2} = \dfrac{1}{2\sqrt{x-2}}$
Zeroes: NONE because the numerator does not vanish.
Critical points: $x = 2$ *as this point belongs to the domain but* $f'(2)$ *DNE.*

We remember this for later: $f'(x) > 0$ *and hence* $f(x)$ *is strictly increasing for* $x > 2$*.*

#3 - Stage 2b*:*

$$f''(x) = (1/2)\,(-1/2)(x-2)^{-3/2} = \frac{-1}{4\sqrt{x-2}^3}$$

$$x > 2 \;\Rightarrow\; 4\sqrt{x-2}^3 > 0 \;\Rightarrow\; \qquad f''(x) < 0$$

Zeroes: NONE because the numerator does not vanish.

We remember this for later: $f''(x) < 0$ and hence $f(x)$ is CD (curves to the right) for $x > 2$.

#3 - Stage 3:

Just as mentioned at the beginning: it's the square root shifted 2 units to the right and it comes as no surprise that the function is increasing and CD on its entire domain. Note that the y-axis cuts the x-axis at $x = 2$.

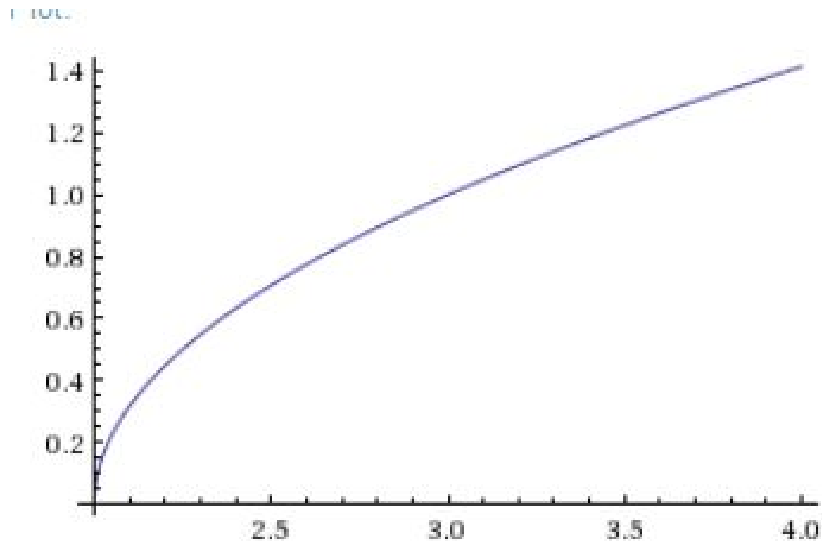

Figure 4: Graph of $f(x) = \sqrt{x-2}$

Exercise 4: *Graph the function* $f(x) = \dfrac{x^2 + 2x - 3}{x + 3}$

Solution to #4:

We can factor $x^2 + 2x - 3 = (x+3)(x-1)$, hence

$$f(x) := \begin{cases} x - 1 & \text{if } x \neq -3, \\ DNE & \text{if } x = -3 \end{cases}$$

We have the straight line $y = x - 1$ with slope 1 and y-intercept -1 with a hole at $x = -3$. The only thing worth mentioning is that -3 is NOT a critical point as it does not belong to the domain $D_f = \{x | x \neq -3\}$.

Exercise 5: *Graph the function* $f(x) = \ln(x + 5)$

Solution to #5:

This is simply the graph of the the function $g(x) = \ln(x)$ shifted 5 units to the left, hence $D_f = (-5, \infty)$
You know the graph of $\ln(x)$ and that the following properties are inherited by $f(x)$:
a) it strictly increases on its entire domain: expect $f'(x) > 0$ on D_f
b) it curves to the right on its entire domain: expect $f''(x) < 0$ on D_f.

Verify on your own that $f'(x) = 1 \cdot \dfrac{1}{x-5} = (x-5)^{-1}$, $f''(x) = (-1)\,(x-5)^{-2} = \dfrac{1}{(x-5)^2}$ and that this in fact implies $f'(x) > 0$ and $f'(x) < 0$ for $x > -5$

The only point to be emphasized is that $x = -5$ is a vertical asymptote for $f(x)$.

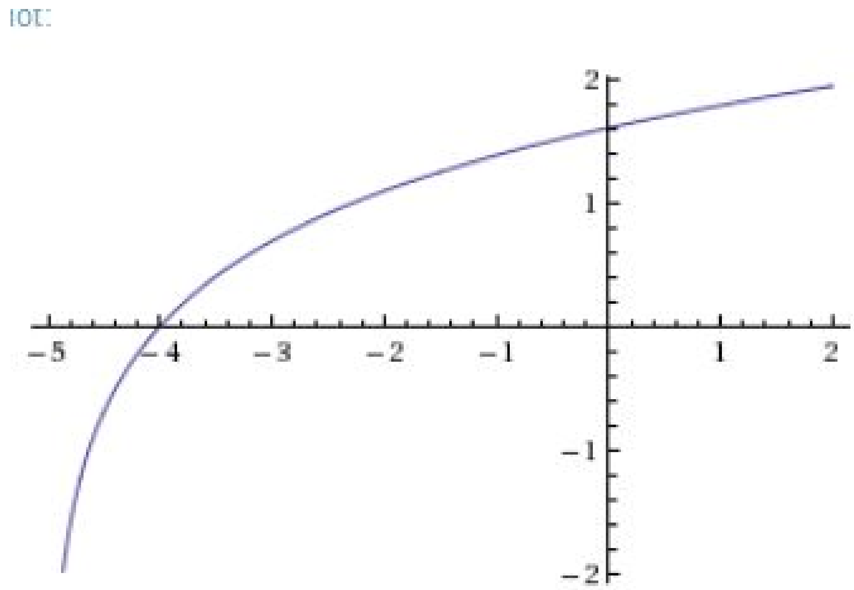

Figure 5: Graph of $f(x) = \ln(x+5)$

Exercise 6: *Graph the function* $f(x) = \dfrac{x^2+6x+5}{x-1}$

Solution to #6:

#6 - Stage 1*:*

Domain: $D_f = \{x | x \neq 1\} = (-\infty, 1) \bigcup (1, \infty)$
y-intercept: $(0, -5)$ *because* $f(0) = 5/(-1) = -5$
x-intercepts: $(-5, 0), (-1, 0)$ *because*

$$x^2 + 6x + 5 = 0 \iff (x+1)(x+5) = 0 \iff x = -5, -1$$

Behavior as $x \to \pm\infty$*:* $\dfrac{x^2+6x+5}{x-1} \approx \dfrac{x^2}{x} = x$ *for huge* $|x|$.

Hence $\lim\limits_{x \to -\infty} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = \infty$.

Hence no horizontal asymptotes.
Vertical asymptotes: $x = 1$ *(we have "12 / zero")*
One-sided limits at $x = 1$*: We have*

$$x \to 1 \Rightarrow x^2 + 6x + 5 \to 12, \quad x \to 1- \Rightarrow x - 1 \to 0-, \quad x \to 1+ \Rightarrow x - 1 \to 0+$$

Hence

$$\lim_{x \to 1-} f(x) = -\infty, \quad \lim_{x \to 1+} f(x) = \infty,$$

#6 - Stage 2a*:*

$$f'(x) = \frac{(2x+6)(x-1) - (x^2+6x+5)}{(x-1)^2} = \frac{2x^2 + 6x - 2x - 6 - x^2 - 6x - 5}{(x-1)^2}$$

$$\Rightarrow \boxed{f'(x) = \frac{x^2 - 2x - 11}{(x-1)^2}}$$

We use the quadratic formula to get the roots (easier to complete the square if you know how):

$$x = \frac{1}{2}\left[2 \pm \sqrt{(-2)(-2) - 4(1)(-11)}\,\right] = 1 \pm \frac{\sqrt{48}}{2} = 1 \pm \frac{\sqrt{16 \cdot 3}}{2} = 1 \pm \frac{4\sqrt{3}}{2}$$
$$\Rightarrow \boxed{x = 1 + 2\sqrt{3} \approx 4.46410,\ 1 - 2\sqrt{3} \approx -2.46410}$$

Function values for the roots of $f'(x)$ are

$$f(1+2\sqrt{3}) = 8 + 4\sqrt{3} \approx 14.9282$$
$$f(1-2\sqrt{3}) = 8 - 4\sqrt{3} \approx 1.0718$$

Critical points (x-values): $1 - 2\sqrt{3},\ 1,\ 1 + 2\sqrt{3}$

To check for plus or minus of the derivative in each of the four intervals separated by the critical points we pick one convenient point in each:

$$x = -3 : f'(x) = \frac{1}{4}\boxed{> 0} \qquad x = 0 : f'(x) = -11\boxed{< 0}$$
$$x = 2 : f'(x) = -11\boxed{< 0} \qquad x = 5 : f'(x) = \frac{1}{4}\boxed{> 0}$$

***#6 - Stage 2b**:*

$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x-11)\cdot 2(x-1)}{(x-1)^4} = \frac{(2x-2)(x-1) - 2(x^2-2x-11)}{(x-1)^3}$$
$$= \frac{2x^2 - 2x - 2x + 2 - 2x^2 + 4x + 22)}{(x-1)^3} = \frac{24}{(x-1)^3} \Rightarrow \boxed{f''(x) = \frac{24}{(x-1)^3}}$$

Zeroes: NONE.
Points for which $f''(x)$ DNE: $x = 1$ (location of the vertical asymptote). To check for plus or minus in each of the two intervals separated by $x = 1$ we pick one convenient point in each. We pick $x = 0, x = 2$.

$$x = 0 : f''(x) = -24\boxed{< 0} \qquad x = 2 : f''(x) = 24\boxed{> 0}$$

***#6 - Stage 3**:*

The points x for which $f'(x) = 0$ or $f''(x) = 0$ (2 for $f'(x)$, none for $f''(x)$) split the domain $D_f = (-\infty, 1)\bigcup(1, \infty)$ into four intervals:

$$\begin{aligned} x < 1 - 2\sqrt{3} &: \quad f'(x) = '+', \quad f''(x) = '-' : && \textit{increasing and CD (curving right)} \\ 1 - 2\sqrt{3} < x < 1 &: \quad f'(x) = '-', \quad f''(x) = '-' : && \textit{decreasing and CD (curving right)} \\ 1 < x < 1 + 2\sqrt{3} &: \quad f'(x) = '-', \quad f''(x) = '+' : && \textit{decreasing and CU (curving left)} \\ x > 1 + 2\sqrt{3} &: \quad f'(x) = '+', \quad f''(x) = '+' : && \textit{increasing and CU (curving left)} \end{aligned}$$

the critical point $(1 - 2\sqrt{3}, 8 - 4\sqrt{3})$ is a local max and the critical point $(1 + 2\sqrt{3}, 8 + 4\sqrt{3})$ is a local min. There are no points of inflection because $f''(x)$ is never zero.

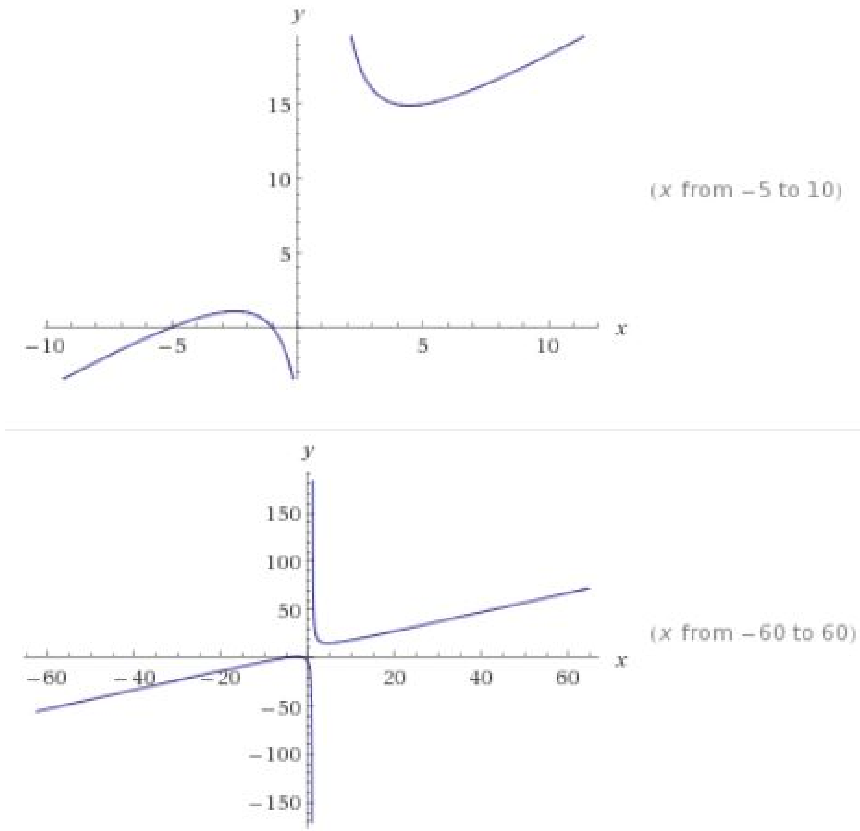

Figure 6: Graph of $f(x) \,=\, \dfrac{x^2+6x+5}{x-1}$

Exercise 7: *Graph the function* $f(x) \,=\, \dfrac{2x^2-6}{x^2+8}$

Solution to #7:

#7 - Stage 1*:*

Domain: $D_f \,=\, (-\infty,\infty)$ *as* x^2+8 *is never zero.*
y-intercept: $(0,-3/4)$ *because* $f(0) = -6/8 = -3/4$
x-intercepts: $(-\sqrt{3},0), (\sqrt{3},0)$ *because*

$$2x^2-6 \,=\, 0 \iff x^2 \,=\, 3 \iff x \,=\, \pm\sqrt{3}$$

Vertical asymptotes: NONE as we never divide by zero.

Behavior as $x \to \pm\infty$*:*

$$\frac{2x^2-6}{x^2+8} \approx \frac{2x^2}{x^2} \,=\, 2 \ \textit{for huge } |x| \ \Rightarrow \ \lim_{x\to-\infty} f(x) \,=\, 2\,, \ \ \lim_{x\to\infty} f(x) \,=\, 2$$

Hence $y=2$ *is a horizontal asymptote.*

#7 - Stage 2a:

$$f'(x) \;=\; \frac{4x(x^2+8)\;-\;(2x^2-6)(2x)}{(x^2+8)^2} \;=\; \frac{4x^3+32x-4x^3+12x}{(x^2+8)^2}$$
$$\Rightarrow \boxed{f'(x) \;=\; \frac{44x}{(x^2+8)^2}}$$

Roots: $f'(x)=0 \iff 44x=0$*, i.e.,* $x=0$*. Function value is* $f(0)=-3/4$
No other critical points as $f'(x)$ *exists for all real numbers.*

To check for plus or minus of the derivative in each of the two intervals separated by the critical point $x=0$ *we pick one convenient point in each:*

$$x \;=\; -1\;:\; f'(x) \;=\; \frac{-44}{65}\boxed{<0} \qquad x\;=\;1\;:\; f'(x) \;=\; \frac{44}{65}\boxed{>0}$$

#7 - Stage 2b:

$$f''(x) \;=\; \frac{44(x^2+8)^2\;-\;44x(2)(x^2+8)(2x)}{(x^2+8)^4} \;=\; \frac{44(x^2+8)}{(x^2+8)^4}\Big((x^2+8)\;-\;(2x)(2x)\Big)$$
$$=\frac{44}{(x^2+8)^3}\Big(-3x^2+8\Big) \;\Rightarrow\; \boxed{f''(x) \;=\; \frac{44(8-3x^2)}{(x^2+8)^3}}$$

Zeroes:

$$f''(x)\;=0 \iff 8-3x^2\;=\;0 \iff x^2=\frac{8}{3} \iff x\;=\;\pm\sqrt{\frac{8}{3}} \;=\;\pm 2\sqrt{\frac{2}{3}} \;=\;\pm 2\,\frac{\sqrt{6}}{3}$$
$$\Rightarrow \boxed{x \;=\;\pm\frac{2\sqrt{6}}{3} \;\approx\; \pm 1.6330}$$

Points for which $f''(x)$ *DNE: NONE. To check for plus or minus in each of the three intervals separated by* $x\pm 2\sqrt{6}/3$ *we pick one convenient point in each. We pick* $x=-2,0,2$*. Observe that the sign of* $f''(x)$ *is the sign of* $8-3x^2$

$$x\;=\;-2\;:8-3\dot{(}-2)^2\;=\;-4\boxed{<0} \qquad x\;=\;0\;: \quad 8-3\dot{0}^2\;=\;8\boxed{>0} \qquad x\;=\;2\;:\;8-3\dot{2}^2(x)\;=\;-44\boxed{<0}$$

#7 - Stage 3:

The points x *for which* $f'(x)=0$ *or* $f''(x)=0$ *split the domain* $D_f=(-\infty,\infty)$ *into four intervals:*

$$\begin{aligned}
x\;<\;-2\sqrt{6}/3\;&:\; f'(x)='-', \quad f''(x)\;=\;'-'\;: && \textit{decreasing and CD (curving right)}\\
-2\sqrt{6}/3<x\;<\;0\;&:\; f'(x)='-', \quad f''(x)\;=\;'+'\;: && \textit{decreasing and CU (curving left)}\\
0<x\;<\;2\sqrt{6}/3\;&:\; f'(x)='+', \quad f''(x)\;=\;'+'\;: && \textit{increasing and CU (curving left)}\\
x\;>\;2\sqrt{6}/3\;&:\; f'(x)='+', \quad f''(x)\;=\;'-'\;: && \textit{increasing and CD (curving right)}
\end{aligned}$$

It follows that $x=0$ *is a local min as we have a horizontal tangent and* $f''(0)>0$.
$-2\sqrt{6}/3$ *is a point of inflection as curvature changes from right to left.*
$2\sqrt{6}/3$ *is a point of inflection as curvature changes from left to right.*

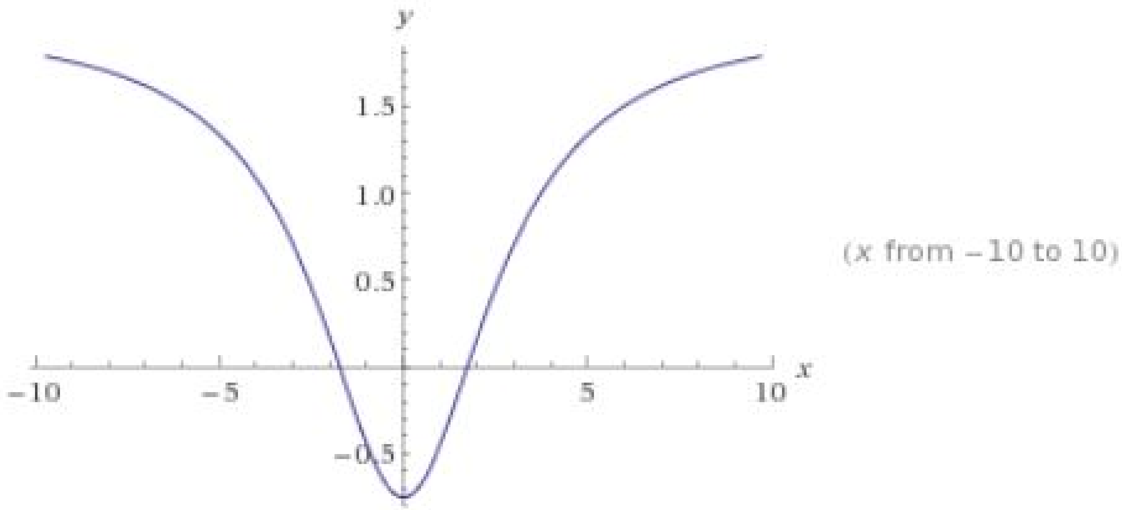

Figure 7: Graph of $f(x) = \dfrac{2x^2-6}{x^2+8}$

The units for x the graph above were chosen rather small so you can see the curvature about the points where $f''(x)$ vanishes. If the range of x included ± 40 you would see more clearly that $y=2$ is a horizontal asymptote.

Exercise 8: *Graph the function* $f(x) = \dfrac{3x+1}{x-2}$

Solution to #8:

#8 - Stage 1*:*

Domain: $D_f = \{x | x \neq 2\} = (-\infty, 2) \bigcup (2, \infty)$
y-intercept: $(0, -1/2)$ *because* $f(0) = 1/(-2) = -1/2$
x-intercept: $(-1/3, 0)$ *because* $f(x) = 0$ *means* $3x+1 = 0$ *or* $x = -1/3$.

Behavior as $x \to \pm\infty$*:*

$$\frac{3x+1}{x-2} \approx \frac{3x}{x} = 3 \text{ for huge } |x| \ \Rightarrow \ \lim_{x\to-\infty} f(x) = 3\,, \ \lim_{x\to\infty} f(x) = 3$$

Hence $y = 3$ is a horizontal asymptote.

Vertical asymptotes: $x = 2$ *(we have "7 / zero")*
One-sided limits at $x = 2$*: We have*

$$x \to 2 \Rightarrow 3x+1 \to 7\,, \quad x \to 2- \Rightarrow x-2 \to 0-\,, \quad x \to 2+ \Rightarrow x-2 \to 0+$$

Hence

$$\lim_{x\to 2-} f(x) = -\infty, \quad \lim_{x\to 2+} f(x) = \infty,$$

#8 - Stage 2a*:*

$$f'(x) = \frac{3(x-2) - (3x+1)}{(x-2)^2} = \frac{3x-6-3x-1}{(x-2)^2} \Rightarrow \boxed{f'(x) = \frac{-7}{(x-2)^2}}$$

There are no roots for the derivative.
Critical points: NONE. $x = 2$ is not a critical point as it is not in D_f.

To check for plus or minus of the derivative in each of the two intervals $(-\infty, 2)$ and $(2, \infty)$ separated by the location $x = 2$ of the vertical asymptote. We pick one convenient point in each. We pick $x = 0, 3$.

$$x = 0 : \; f'(x) \; = \; -7/4 \boxed{< 0} \qquad x \; = \; 3 : \; f'(x) \; = \; -7/1 \boxed{< 0}$$

This means that $f(x)$ is strictly decreasing everywhere.

***#8 - Stage 2b**:*

$$f''(x) \; = \; \frac{0(x-2)^2 \; - \; (-7)(2)(x-2)}{(x-2)^4} \; = \; \frac{0-(-14)}{(x-2)^3} \; \Rightarrow \; \boxed{f'(x) \; = \; \frac{14}{(x-2)^3}}$$

There are no zeroes for the second derivative and the only x-value where concavity might change is the vertical asymptote. We pick again $x = 0, 3$.

$$x = 0 : \; f''(x) \; = \; 14/(-8) \boxed{< 0} \qquad x \; = \; 3 : \; f''(x) \; = \; 14/1 \boxed{> 0}$$

***#8 - Stage 3**:*

The only point x for which $f'(x) = 0$ or $f''(x) = 0$ is $x = 2$ and we only need to examine two intervals:

$x \; < \; 2 :$ $f'(x) =' -'$, $f''(x) \; = \; '-' :$ *decreasing and CD (curving right)*
$x \; > \; 2 :$ $f'(x) =' -'$, $f''(x) \; = \; '+' :$ *decreasing and CU (curving left)*

It follows that there are no minima or maxima or points of inflection.

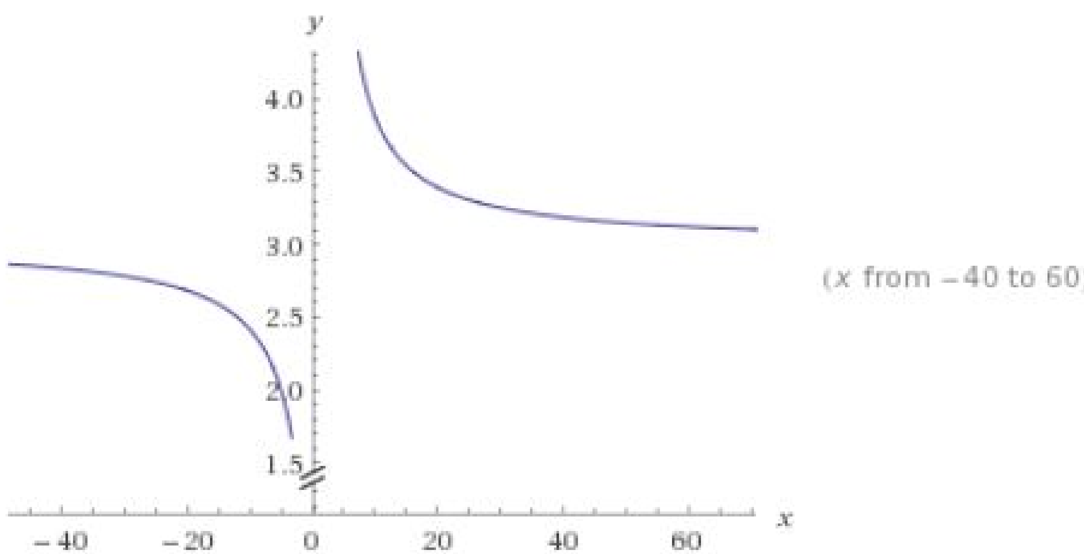

Figure 8: Graph of $f(x) \; = \; \dfrac{3x+1}{x-2}$

***Exercise 9:** Graph the function* $f(x) \; = \; -2x^3 - 9x^2 + 108x - 10$

Solution to #9:

#9 - Stage 1:

Domain: all real numbers as $f(x)$ is a polynomial.
y-intercept: $f(0) = -10$
x-intercepts: too difficult to compute $f(x) = 0$ with pen and paper but easy with Wolfram Alpha or other math software: Solutions are $x_1 \approx -9.9678$, $x_2 \approx .093334$, $x_3 \approx 5.3744$.
Horizontal asymptotes: None: No polynomial has horizontal asymptotes unless it is of the form $f(x) = const$.
Behavior as $x \to \pm\infty$: $f(x) \approx -2x^3$ for huge $|x|$. Hence $\lim\limits_{x\to-\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = -\infty$.
Vertical asymptotes: None and no other kind of discontinuity as $f(x)$ is a polynomial.

#9 - Stage 2a:

$$f'(x) = -6x^2 - 18x + 108 = -6(x^2 + 3x - 18) = -6(x+6)(x-3)$$

Zeroes: $f'(x) = 0 \iff (x+6)(x-3) = 0$, i.e., $\boxed{x = -6}$ or $\boxed{x = 3}$
Critical points: None besides $x = -6, x = 3$ as there are no "problem points" for the polynomial $f'(x)$. To check for plus or minus in each of the three intervals separated by the critical points we pick one convenient point in each. We pick $x = -7, 0, 4$.

$x = -7 : f'(x) = (-6)(-1)(-10)\boxed{< 0}$ $\quad x = 0 : f'(x) = 108\boxed{> 0}$ $\quad x = 4 : f'(x) = (-6)(10)(1)\boxed{< 0}$

#9 - Stage 2b:

$f''(x) = -12x - 18$
Zeroes: $f''(x) = 0 \iff 12x = -18$, i.e., $\boxed{x = -3/2}$
Points for which $f''(x)$ DNE: None because there are no "problem points for the polynomial $f''(x)$. To check for plus or minus in each of the two intervals separated by $x = -3/2$ we pick one convenient point in each. We pick $x = -2$ and $x = 0$.

$$x = -2 : f''(x) = 24 - 18\boxed{> 0} \qquad x = 0 : f''(x) = -18\boxed{< 0}$$

#9 - Stage 3:

The points x for which $f'(x) = 0$ or $f''(x) = 0$ split the domain $D_f = (-\infty, \infty)$ into four intervals:

$$\begin{aligned}
x < -6 :&\quad f'(x) = '-', \quad f''(x) = '+' : &&\textit{decreasing and CU (curving left)}\\
-6 < x < -3/2 :&\quad f'(x) = '+', \quad f''(x) = '+' : &&\textit{increasing and CU (curving left)}\\
-3/2 < x < 3 :&\quad f'(x) = '+', \quad f''(x) = '-' : &&\textit{increasing and CD (curving right)}\\
x > 3 :&\quad f'(x) = '-', \quad f''(x) = '-' : &&\textit{decreasing and CD (curving right)}
\end{aligned}$$

It follows that $x = 13/2$ is a point of inflection as curvature changes from left to right, the critical point -6 has a local min and the critical point 3 has a local max.

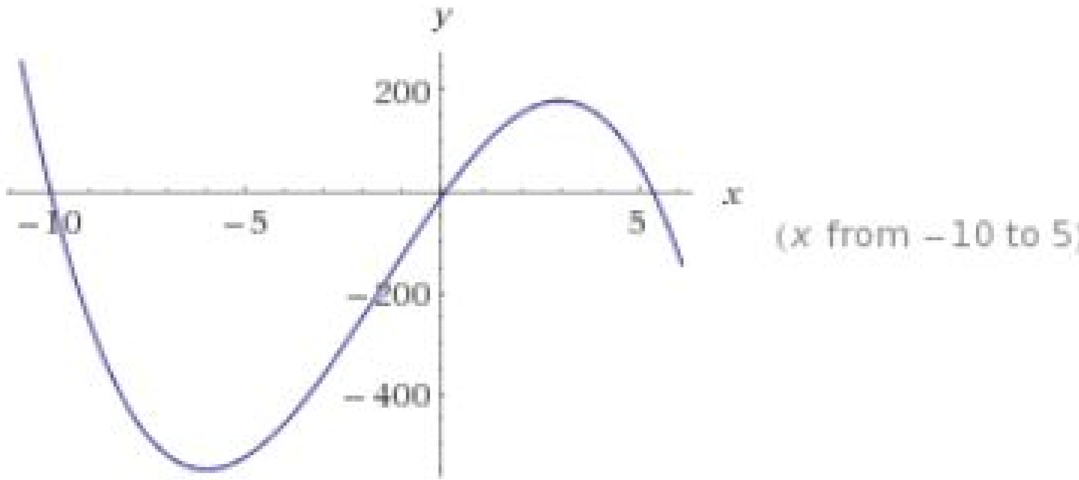

Figure 9: Graph of $f(x) = -2x^3 - 9x^2 + 108x - 10$

Exercise 10: *Graph the function* $f(x) = \dfrac{4}{x^2+5}$. *The first and second derivative have already been computed for you:*

$$f'(x) = \frac{-8x}{(x^2+5)^2} \qquad f''(x) = \frac{24x^2-40}{(x^2+5)^3}$$

Solution to #10:

#10 - Stage 1*:*

Domain: all real numbers as $f(x)$ is a rational function for which the denominator is never zero. It follows that $f(x)$ is continuous everywhere.
y-intercept: $f(0) = 4/5$
x-intercepts: NONE as the numerator is never zero.

Vertical asymptotes: None as the denominator is never zero.
Behavior as $x \to \pm\infty$: $f(x) \approx 0$ for huge $|x|$. Hence $\lim\limits_{x\to-\infty} f(x) = 0\,, \quad \lim\limits_{x\to\infty} f(x) = 0\,.$
Hence the x-axis is a horizontal asymptotes.

#10 - Stage 2a*:*

Zeroes of $f'(x) = \dfrac{-8x}{(x^2+5)^2}$*:*
$f'(x) = 0 \iff -8x = 0$*, i.e.,* $\boxed{x = 0}$
Critical points: None besides $x = 0$ as there are no "problem points" where the denominator of $f'(x)$ vanishes. To check for plus or minus in each of the two intervals separated by the critical point we pick one convenient point in each. We pick $x = -1, 1$.

$$x = -1\ :\ f'(x) = 8/36)\boxed{>0} \qquad x = 1\ :\ f'(x) = -8/36\boxed{<0}$$

#10 - Stage 2b*:*

Zeroes of $f''(x) = \dfrac{24x^2-40}{(x^2+5)^3} = \dfrac{8(3x^2-5)}{(x^2+5)^3}$*:*

$f''(x) = 0 \iff 3x^2 - 5$, *i.e.,* $\boxed{x = \pm\sqrt{5/3}}$ *i.e.,* $x \approx \pm 1.301$
Points for which $f''(x)$ *DNE: None because its denominator never vanishes. To check for plus or minus in each of the three intervals separated by* $x = \pm\sqrt{5/3}$ *we pick one convenient point in each. We pick* $x = -2, 0, 2$.

$$x = -2 : f''(x) = \frac{96-40}{9^3}\boxed{>0} \quad x = 0 : f''(x) = -40/5^3\boxed{<0} \quad x = 2 : \quad f''(x) = \frac{96-40}{9^3}\boxed{>0}$$

#10 - Stage 3*:*

The points x *for which* $f'(x) = 0$ *or* $f''(x) = 0$ *split the domain* $D_f = (-\infty, \infty)$ *into four intervals:*

$$\begin{aligned}
x < -\sqrt{5/3} : &\quad f'(x) = '+', \quad f''(x) = '+' : \quad \textit{increasing and CU (curving left)}\\
-\sqrt{5/3} < x < 0 : &\quad f'(x) = '+', \quad f''(x) = '-' : \quad \textit{increasing and CD (curving right)}\\
0 < x < \sqrt{5/3} : &\quad f'(x) = '-', \quad f''(x) = '-' : \quad \textit{decreasing and CD (curving right)}\\
x > \sqrt{5/3} : &\quad f'(x) = '-', \quad f''(x) = '+' : \quad \textit{decreasing and CU (curving left)}
\end{aligned}$$

It follows that $x = \pm\sqrt{5/3}$ *are points of inflection as concavity changes. the critical point* 0 *has a local max as* $f''(0) = -40/5^3 < 0$.

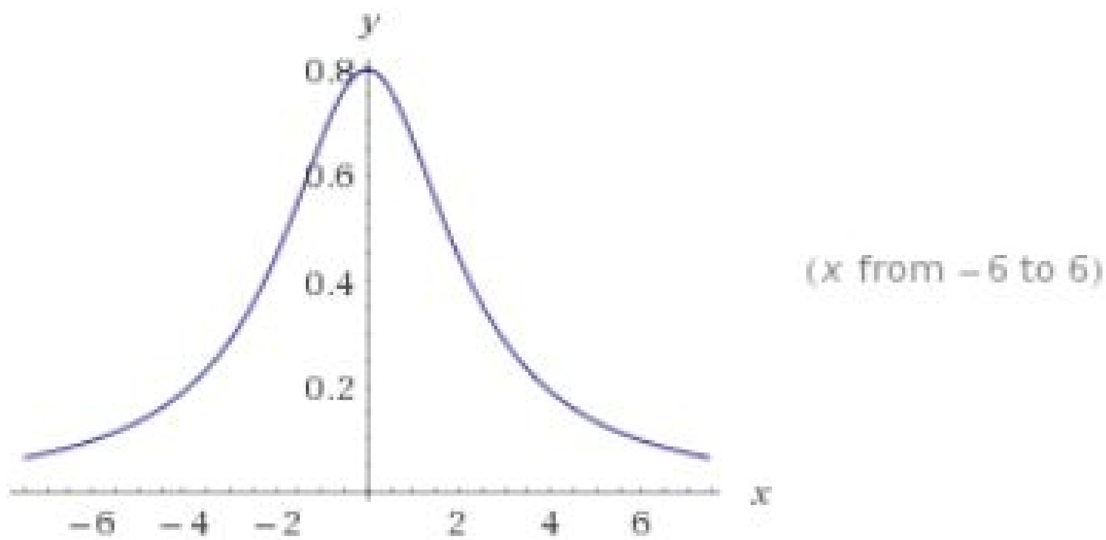

Figure 10: Graph of $f(x) = \dfrac{4}{x^2+5}$

Exercise 11: *Graph the function* $f(x) = \dfrac{x^2}{x+6}$. *The first and second derivative have already been computed for you:*

$$f'(x) = \frac{x^2+12x}{(x+6)^2} \qquad f''(x) = \frac{72}{(x+6)^3}$$

Solution to #11:

#11 - Stage 1*:*

Domain: $D_f = \{x | x \neq -6\} = (-\infty, -6) \bigcup (-6, \infty)$
y-intercept: $(0,0)$ *because* $f(0) = 0/6 = 0$
x-intercept: $(-6,0)$ *because* $f(x) = 0 \iff x+6 = 0 \iff x = -6$
Behavior as $x \to \pm\infty$*:* $\dfrac{x^2}{x+6} \approx x$ *for huge* $|x|$*.*

Hence $\lim\limits_{x\to-\infty} f(x) = -\infty\,,\quad \lim\limits_{x\to\infty} f(x) = \infty\,.$

Hence no horizontal asymptotes.
Vertical asymptotes: $x = -6$ *(we have "36 / zero")*
One-sided limits at $x = -6$*: We have*

$$x \to -6 \Rightarrow x^2 \to 36\,, \quad x \to -6- \Rightarrow x+6 \to 0-\,, \quad x \to -6+ \Rightarrow x+6 \to 0+$$

Hence

$$\lim_{x\to-6-} f(x) = -\infty, \quad \lim_{x\to-6+} f(x) = \infty,$$

#11 - Stage 2a*:*

$$f'(x) = \frac{x^2+12x}{(x+6)^2} = 0 \iff x^2+12x = 0 \iff x(x+12) = 0 \iff \boxed{x = -12, x = 0}$$

Function values for the roots of $f'(x)$ *are*

$$f(0) = 0\,, \qquad f(-12) = 12\cdot 12/18 = 3\cdot 4\cdot 6\cdot 2/(3\cdot 6) = 4\cdot 2 = 8$$

Critical points: None besides $x = -12, 0$ *but we must watch out for the sign of* $f'(x)$ *as we cross the vertical asymptote at* $x = -6$*.*

To check for plus or minus of the derivative in each of the four intervals separated by $x = -12, -6, 0$ *we pick one convenient point in each. As the denominator is always positive, the sign of* $f'(x)$ *equals that of the numerator which we factor as* $x(x+12)$*:*

$$x = -13 : x(x+12) = (-13)(-1)\boxed{>0} \qquad x = -7 : x(x+12) = (-7)(5)\boxed{<0}$$
$$x = -1 : x(x+12) = (-1)(11)\boxed{<0} \qquad x = 1 : x(x+12) = 1(13)\boxed{>0}$$

#11 - Stage 2b*:*

$f''(x) = \dfrac{72}{(x+6)^3}$ *has no zeroes as the numerator never vanishes.*
Points for which $f''(x)$ *DNE:* $x = -6$ *(location of the vertical asymptote). To check for plus or minus to the left and to the right of that point we pick one convenient point in each. We pick* $x = -7, -5$*.*

$$x = -7 : f''(x) = \frac{72}{(-1)^3}\boxed{<0} \qquad x = -5 : f''(x) = \frac{72}{(1)^3}\boxed{>0}$$

#11 - Stage 3*:*

The points x *for which* $f'(x) = 0$ *or* $f''(x) = 0$ *(2 for* $f'(x)$*, none for* $f''(x)$*) together with* $x = -6$ *split the domain* $D_f = (-\infty, -6) \bigcup (-6, \infty)$ *into four intervals:*

$$\begin{aligned}
x < -12 &: \; f'(x) = '+', \quad f''(x) = '-' : && \textit{increasing and CD (curving right)}\\
-12 < x < -6 &: \; f'(x) = '-', \quad f''(x) = '-' : && \textit{decreasing and CD (curving right)}\\
-6 < x < 0 &: \; f'(x) = '-', \quad f''(x) = '+' : && \textit{decreasing and CU (curving left)}\\
x > 0 &: \; f'(x) = '+', \quad f''(x) = '+' : && \textit{increasing and CU (curving left)}
\end{aligned}$$

the critical point $(-12, 8)$ is a local max and the critical point $(0, 0)$ is a local min. There are no points of inflection because $f''(x)$ is never zero.

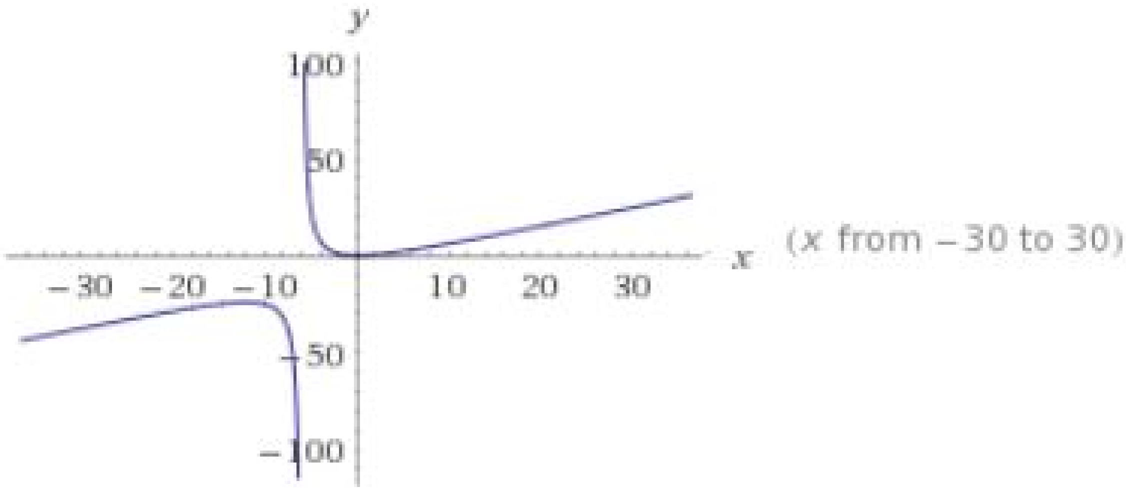

Figure 11: Graph of $f(x) = \dfrac{x^2}{x+6}$