Math 220 - Calculus f. Business and Management - Worksheet 13

Solutions for Worksheet 13 - Definition and geometric interpretation of derivatives

Equations for a line

Exercise 1: *Find three forms of the equation for the line connecting the listed pairs of points. The forms are* ***point-slope*** *($y - y_0 = m(x - x_0)$),* ***slope intercept*** *($y = mx + b$) and* ***general form*** *($px + qy + r = 0$).*

$$\textbf{\textit{1a}} : (1,1) \text{ and } (4,64), \qquad \textbf{\textit{1b}} : \ (2,8) \text{ and } (4,64), \qquad \textbf{\textit{1c}} : \ (3,27) \text{ and } (4,64).$$

Notice that each pair of points is on the graph of $f(x) = x^3$.

Solution for #1*: We give one solution for each of the above three pairs of points.*

We note that in each case $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Solution to 1a*, given in point-slope form:*

$$y - 1 = \frac{64 - 1}{4 - 1}(x - 1) = 21(x - 1).$$

Solution to 1b*, given in slope-intercept form: Point-slope form is*

$$y - 8 = \frac{64 - 8}{4 - 2}(x - 2) = 28(x - 2).$$

The intercept b corresponds to $x = 0$: $b - 8 = 28(-2) \rightsquigarrow b = -48$ and slope $m = \Delta y/\Delta x = 56/2 = 28$. Hence slope-intercept form is

$$y = mx + b = 28x - 48.$$

Solution to 1c*, given in general form: We get it from point slope form as follows:*

$$y - 27 = \frac{64 - 27}{4 - 3}(x - 3) = 37(x - 3) = 37x - 111 \rightsquigarrow y - 37x + 84 = 0.$$

Slope of the tangent to a curve

Exercise 2: *Find the slope of the line tangent to each curve at the given value of x.*

$$\textbf{\textit{2a}} : f(x) = 2x^2 + x - 3 \quad \text{at } x = 4,$$

$$\textbf{\textit{2b}} : f(x) = \begin{cases} 4x + 2 & \text{if } x < -2, \\ 3x & \text{if } x > -2 \end{cases} \quad \text{at } x = -3, -2, -1.$$

Solution to 2a*:*

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{2x^2+4hx+2+x+h-3-2x^2-x+3}{h} \\
&= \frac{4hx+2h^2+h}{h} \;=\; \frac{h(4x+2h+1)}{h} \;=\; 4x+1+2h \\
&\rightsquigarrow \quad f'(x) = \lim_{h\to 0}\big(4x+2h+1\big) \;=\; 4x+1
\end{aligned}
$$

Plug in $x=4$ and you get $f'(4)=16+1=17$.

Solution to 2b:

Let us first examine the case $x=-2$. This point is not in the domain. Hence

$$
f(-2) \text{ DNE} \;\rightsquigarrow\; f(-2+h)-f(-2) \text{ DNE} \;\rightsquigarrow\; \frac{\Delta y}{\Delta x} = \frac{f(-2+h)-f(-2)}{h} \text{ DNE} \;\rightsquigarrow\; f'(-2) \text{ DNE}.
$$

For all points $x \neq -2$:

$$
\begin{aligned}
x<-2 \rightsquigarrow \quad \frac{f(x+h)-f(x)}{h} &= \frac{4x+4h+2-4x-2}{h} = \frac{4h}{h} \;=\; 4 \quad \rightsquigarrow \quad f'(x)=4, \\
x>-2 \rightsquigarrow \quad \frac{f(x+h)-f(x)}{h} &= \frac{3x+3h-3x}{h} = \frac{3h}{h} \;=\; 3 \quad \rightsquigarrow \quad f'(x)=3.
\end{aligned}
$$

It follows that $f'(-3)=4$ and $f'(-1)=3$.

Derivatives

Exercise 3: *Find the derivative of each function. Then use it to find the line tangent to the curve at the given value of x.*

$$
\begin{aligned}
\textbf{3a}: f(x) &\;=\; \sqrt{3x-12} \quad \text{at } x=2,\; x=4,\; x=6, \\
\textbf{3b}: f(x) &\;=\; \frac{7}{2x} \quad \text{at } x=0,\; x=5.
\end{aligned}
$$

Solution to 3a:

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{3x+3h-12}-\sqrt{3x-12}}{h} \\
&= \frac{\big(\sqrt{3x+3h-12}-\sqrt{3x-12}\big)\cdot\big(\sqrt{3x+3h-12}+\sqrt{3x-12}\big)}{h\big(\sqrt{3x+3h-12}+\sqrt{3x-12}\big)} \\
&= \frac{3x+3h-12-(3x-12)}{h\big(\sqrt{3x+3h-12}+\sqrt{3x-12}\big)} \;=\; \frac{3h}{h\big(\sqrt{3x+3h-12}+\sqrt{3x-12}\big)} \\
&= \frac{3}{\sqrt{3x+3h-12}+\sqrt{3x-12}}.
\end{aligned}
$$

If $h\to 0$ (h approaches 0) then $\sqrt{3x+3h-12}\to\sqrt{3x-12}$, i.e.,

$$
f'(x) \;=\; \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \;=\; \lim_{h\to 0}\frac{3}{\sqrt{3x+3h-12}+\sqrt{3x-12}} \;=\; \frac{3}{2\sqrt{3x-12}}.
$$

Let's compute some tangent lines:

$$x = 2 \rightsquigarrow \quad 3x - 12 \;=\; -6 \;< 0 \;\rightsquigarrow\; f(2), f'(2) \textit{ DNE } \rightsquigarrow \textit{ no tangent}$$
$$x = 4 \rightsquigarrow \quad 3x - 12 \;=\; 0 \;\rightsquigarrow\; f'(4) \textit{ DNE } \rightsquigarrow \textit{ no tangent}$$
$$x = 6 \rightsquigarrow \quad f(6) \;=\; \sqrt{6},\; f'(6) \;=\; \frac{3}{2\sqrt{6}} \;\rightsquigarrow\; y - \sqrt{6} \;=\; \frac{3}{2\sqrt{6}}(x-6)$$

is the point-slope form of the tangent at $(6, f(6))$.

Solution to 3b:

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \cdot \left(\frac{7}{2(x+h)} - \frac{7}{2x} \right) \;=\; \frac{7}{2h} \cdot \frac{x - (x+h)}{(x+h)x} \;=\; \frac{7}{2} \cdot \frac{-1}{(x+h)x}.$$

If $h \to 0$ *(*h *approaches* 0*) then* $-1/\big((x+h)x\big) \to -1/x^2$*, i.e.,*

$$f'(x) \;=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h \to 0} \frac{7}{2} \cdot \frac{-1}{(x+h)x} \;=\; \frac{-7}{2x^2}.$$

Tangent lines:

$$x = 0 \rightsquigarrow \quad f(2), f'(2) \textit{ DNE } \rightsquigarrow \textit{ no tangent}$$
$$x = 5 \rightsquigarrow \quad f(5) \;=\; 7/10,\; f'(5) \;=\; \frac{-7}{50} \;\rightsquigarrow\; y - 7/10 \;=\; \frac{-7}{50}(x-5)$$

is the point-slope form of the tangent at $(5, f(5))$.